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ADAPTIVE FINITE ELEMENT COMPUTATIONS OF COMPLEX FLOWS

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SUMMARY

Recent efforts to develop and apply adaptive finite element techniques for solving complex flow problems are reviewed. The emphasis is not on new methods but on how to use existing methods to achieve accurate predictions. Various sources of errors, and means of detecting them, are discussed. The aim is to identify various sources of errors and their effects on adaptivity. For simplicity and clarity, examples are taken from steady state two-dimensional flow problems. The paper discusses the use of adaptive techniques for achieving grid-independent solutions. Wherever possible predictions are compared with measurements. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: complex flow; adaptive techniques; finite element

1. INTRODUCTION

Accurate predictions of complex flows have been the subject of active research for many years and much progress has been achieved through developments in discretizations, solvers and adaptivity. The latter offers the potential of producing solutions with controlled accuracy in a cost-effective manner. Many techniques have been proposed and studied from both theoretical and practical points of view. However, adaptive methods have yet to find their niche in general application work. This is due in part to the fact that adaptivity is a relatively recent technology. Much remains to be done to assess it more thoroughly. It is also fair to say that, until now, techniques were developed successfully for rather restricted classes of problems or for very specific numerical schemes. Adaptive methods will likely have a more pronounced impact on computations and simulations after they have matured to the point where they are general enough to cover the scope of general purpose solvers offered by software vendors.

This paper provides a review of the authors work over the past years in developing adaptive finite element algorithms applicable to a broad class of problems. The adaptive technology is composed of three key elements: the flow solver, the error estimator and the mesh adaptation technique. Focus has been mostly on error estimation techniques and their application to non-trivial multi-field flow problems: laminar isothermal flows, heat transfer by free, mixed and forced convection, conjugate heat transfer, flows with variable fluid properties, two-equation models for turbulent flow and heat transfer, and more recently, compressible

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Table I. Flow over a backward facing step

Author	Turbulence model	L/H
Kim *et al.*	Experiment	7 ± 0.5
Mansour and Morel	k–ϵ	5.2
Pollard	k–ϵ	5.88
Rodi *et al.*	k–ϵ	5.8
Launder *et al.*	ASM	6.9
Abdelmeguid *et al.*	k–ϵ	6
Demirdzic *et al.*	Modified k–ϵ	6.2
Donaldson *et al.*	RSM	6.1
Ilegbusi and Spalding	Modified k–ϵ	7.2
Nallasamy and Chen	k–ϵ	5.8
Syed *et al.*	k–ϵ	5.8
Ilinca and Pelletier	k–ϵ	6.2

turbulent flows. The emphasis of this paper and the work of the authors is not so much on new techniques, but rather on how to use existing methods to consistently deliver reliable results. Some effort has been spent on identifying and understanding the various sources of errors and their effects on adaptivity.

Two examples of flows for which adaptive methods can provide quality control of the solutions to insure some level of accuracy and reliability of predictions are provided. Table I presents results for the length of the recirculation zone for the experimental condition of Kim *et al.* [1] as reported by Nallasamy [2]. All authors use a variant of the $k-\epsilon$ model, wall functions and a TEACH-type solution algorithm. The only exceptions are the prediction of Donaldson who used a Reynolds stress model, and that of Ilinca and Pelletier [13] obtained with an adaptive finite element method. As can be seen there are significant discrepancies between the predictions. The only possible causes for these differences are the meshes used and details of the computer implementation.

The scatter between predictions is even worse for turbulent heat transfer [3]. Table II presents the maximum Nusselt number downstream of a sudden pipe expansion. The lowest predicted value is in error by 50%, while the highest prediction is in error by more than 100%. According to Launder, the main source of error is the near-wall model. The second source of

Table II. Maximum Nusselt number downstream of a sudden pipe expansion

Author	$Nu_{\max}$
Numerical #1	1660
Numerical #2	375
Numerical #3	1205
Numerical #4	1330
Numerical #5	915
Numerical #6	2036
Numerical #7	574
Numerical #8	1440
Numerical #9	921
Numerical #10	943
Numerical #11	975
Experimental	932

error is numerical: mesh size and artificial diffusion owing to upwind discretization of convective terms.

While turbulence modelling issues are still a topic of hot debate, numerical and discretization issues can now be addressed in a rigorous and systematic manner so as to minimize their impact on the uncertainty of predictions. Adaptive methods are a powerful tool to control numerical errors and to obtain 'numerically exact' solutions to the differential equations. In this way mathematical modelling issues can be studied and evaluated with more confidence.

The paper is organized as follows. A general framework for adaptivity is first reviewed. Error estimations techniques are then described from the point of view of their generality and applicability to complex flows. Applications are then presented which cover incompressible turbulent flow, turbulent heat transfer and tansonic turbulent flow.

2. ADAPTIVITY

A general adaptive algorithm takes the following form [4]:

1. Define an initial discretization: (M_0, P_0);
2. Solve the discretized equations. The result is the numerical solution: U_i;
3. Compute the error estimate: $E_i(M_i, P_i, U_i)$;
4. If the E_i is smaller than a pre-established tolerance ϵ, the computation is stopped;
5. Otherwise the discretization is adapted, i.e. a new discretization is derived from previous ones: $(M_{i+1}, P_{i+1}) = A(M_j, P_j, U_j)$, $j = 1, \ldots, i$;
6. Go to step 2.

A discretization consists of a mesh and a polynomial approximation (M_i, P_i), where M_i is the mesh at the ith cycle of adaptation and P_i is the distribution of polynomial interpolation over the mesh. The exact solution is denoted by U_{ex} and its finite element approximation by U_{i}. The true error is defined as $e_i = U_{\text{ex}} - U_i$. Finally, E_i is the estimate of e_i. This general framework accommodates most techniques found in the literature: h-, p-, or r-methods, and combinations such as hp-techniques. According to Babuska, the technique is termed adaptive if and only if the above iteration converges [4]. Thus, adaptivity has much in common with optimal control of solution accuracy. Depending on how convergence is measured, different kinds of adaptivity are obtained. For example, if

$$\lim_{i \to \infty} \| U_{\text{ex}} - U_i \| = 0,$$

adaptivity is achieved with respect to convergence: accuracy will improve with mesh refinement. If, in addition,

$$\lim_{i \to \infty} \theta_i = \lim_{i \to \infty} \frac{E_i}{e_i} = 1, \tag{1}$$

we have adaptivity with respect to the efficiency index θ of the error estimator. In this case not only will the solution improve with mesh refinement, but the accuracy of the error estimate also improves with refinement. This implies that the error estimate will be a good approximation of the error.

In this paper, H-remeshing is considered because it is particularly well-suited to steady state problems and provides great control over the local mesh density. The remeshing strategy follows that presented by Peraire [5]. Given a desired relative accuracy of η%, the target error over the whole domain is defined as

$$\|E\|_\Omega = \eta \, \|U_i\|_\Omega,$$

where the subscript Ω indicates that the norm is computed over the whole domain. A target accuracy over an element is computed as

$$e_T = \eta \, \frac{\|U\|}{\sqrt{n_e}},$$

where e_T is the element target error and n_e is the number of elements in the current mesh. To determine the new mesh size, it is required that the improved mesh be optimal, i.e. the norm of the error is the same for all elements in the mesh. The error can be related to the element size through the asymptotic rate of convergence of the finite element method. On element k we can write:

$$e_k = Ch_k^p,$$

where p is the asymptotic rate of convergence of the finite element scheme. The proportionality constant C is problem-dependent and usually unknown. A similar expression can be written for the target error and the ideal element size, as:

$$e_T = C\delta_k^p.$$

These two equations can be solved for the local mesh size, leading to

$$\delta_k = \left\{ \frac{\eta \, \|U\|^p}{\sqrt{n_e} \, \|e_k\|} \right\}^{1/p} h_k.$$

There still remains to be discussed the process by which the error estimate is obtained.

3. ERROR ESTIMATION

3.1. Generalities

There are several techniques for computing estimates given a finite element solution. A recent review is provided by Ainsworth and Oden [6] who consider interpolation error estimators, explicit residual estimators, implicit residual techniques, and projection or gradient recovery methods.

The challenge is to find at least one approach capable of handling most if not all of the following characteristics common to many complex flows: compressible and incompressible flows, laminar and turbulent flow regimes, Newtonian and non-Newtonian fluids, variable fluid properties, non-linearities, flows at high Reynolds, Péclet and Rayleigh numbers, thin features, and upwinding or stabilization techniques. The last one is required for the majority of flow problems and proves to be a major obstacle to the practical application of a variety of error estimators.

An ideal estimator would have the following characteristics:

- The estimator must have the *scope* of the flow solver. This means that it must be applicable to the spectrum of flows that the solver can handle.
- Its behaviour must be predictable.
- It must account for the various sources of errors.
- It must be mathematically correct.
- It must be rooted in the physics of the flow.
- It must not be overly expensive to compute.

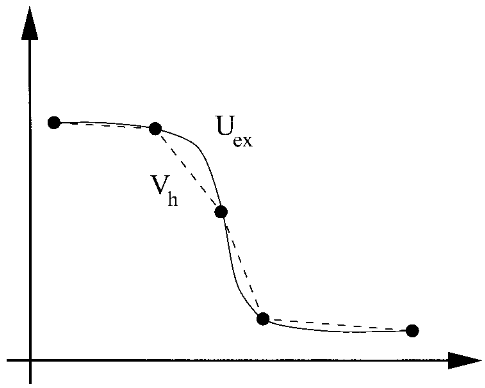

Figure 1. Interpolation error.

There are various sources of errors. Interpolation errors are due to the polynomial approximation of the solution assuming exact nodal values; see Figure 1. Discretization error is the additional error due to use of the finite element method to generate nodal values of the solution; see Figure 2. Most error estimation techniques attempt to measure interpolation and/or on discretization errors. However, they often neglect other error sources arising from approximate representation of the data: geometry, boundary conditions or physical properties of the fluid. Geometric errors usually result from the approximation of complex curves by linear segments. Boundary condition errors result from polynomial approximation of complicated functions defined on the boundary. They can result in improper mass flow or energy inputs to the system.

Blain *et al.* present a striking example of the effect of data error on the global accuracy of storm surge predictions [7]. They have shown that accurate representation of water depth is critical. The mesh must be refined near the edge of the continental shelf because of rapid variations in a source term, even though the flow is smooth. Another example is the need for proper resolution of the variation of the eddy viscosity in two-equation models of turbulence. For the $k-\epsilon$ model, the eddy viscosity is given by

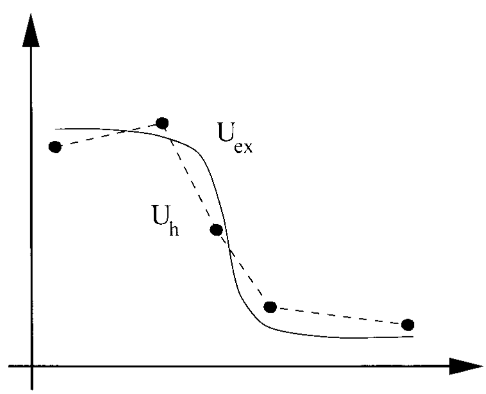

Figure 2. Discretization error.

$$\mu_T = \rho C_\mu k^2/\epsilon.$$

Consider the very simple situation where k increases linearly while ϵ decreases linearly. Then μ_T is a rational function that may present rapid variations even if k and ϵ are represented exactly by linear elements. Note that μ_T is the sole mechanism for transport of momentum heat and turbulence kinetic energy by turbulence fluctuations in the two-equation models of turbulence. Hence, accurate predictions of turbulent flows require a good discretization of the eddy viscosity.

3.2. Error estimators

For simplicity, consider the simple case of a Poisson equation:

$$-\nabla^2 U = f.$$

Explicit residual estimators are the most economical because they only require the computation of element residuals:

$$\|E\|^2 \le C\left\{h_k^2\|\nabla^2 U_h - f\|_K + h_k\left\|\left[\!\left[\frac{\partial U_h}{\partial n}\right]\!\right]\right\|_b\right\},$$

where the first term on the right-hand-side is the element residual, a measure of the accuracy of the solution inside an element. The second term measures the accuracy of the solutions across element faces. The symbol $[\![F]\!]$ denotes the jump of F across an element face. The technique is simple. However, it is valid only for elliptic problems. Moreover, the value of the constant C is unknown and its approximation is a non-trivial task. At this time, it appears that the theory is insufficient for convection-dominated flows. An application to creeping flow of polymers can be found in [8].

Implicit residual estimators measure the error by solving a local finite element problem for the error. For the Poisson equation, the finite element weak form is given by

$$\int_K \nabla U \cdot \nabla V \, \mathrm{d}A = \int_K fV \, \mathrm{d}A + \oint_{\partial K} \frac{\partial U}{\partial n} \cdot V \, \mathrm{d}S.$$

Substitution of $e = U_{ex} - U_h$ in the above equation leads to

$$\int_K \nabla e \cdot \nabla V \, \mathrm{d}A = \int_K [f + \nabla^2 U_h] V \, \mathrm{d}A + \oint_{\partial K} \frac{1}{2}\left[\!\left[\frac{\partial U_h}{\partial n}\right]\!\right] \mathrm{d}S. \tag{2}$$

Again, the first term on the right-hand-side measures the quality of the solution in the element, while the second one measures the accuracy of the solution across element faces. The local problem (1) is solved separately on each element using basis functions of higher order that those used to solve Equation (1). This results is a small linear system for each element.

This approach was successfully generalized to a variety of laminar flows and to mixing length models of turbulent flows. For the case of two-dimensional flow with heat transfer, the local problem reads

$$\begin{bmatrix} K_U & C & B \\ C^T & 0 & 0 \\ D & 0 & K_T \end{bmatrix} \begin{Bmatrix} e_U \\ e_P \\ e_T \end{Bmatrix} = \begin{Bmatrix} r(U_h) + \text{Jump} \\ \nabla \cdot U_h \\ r(T_h) + \text{Jump} \end{Bmatrix},$$

where e_U, e_P and e_T denote the velocity, pressure and temperature errors respectively while $r(X)$ denotes the element residual for variable X, and Jump represents the diffusion flux discontinuity across element faces. The technique works very well for Galerkin discretizations of the equations. However, for convection dominated flows, some form of upwinding is required in the flow solver. This is especially true for two-equation models of turbulent flows. In such cases, the implicit residual estimator performed poorly. It seems that an appropriate local problem cannot be obtained by direct substitution of the error definition as done above.

Projection error estimators were introduced by Zienkiewicz and Zhu [9,10]. The approach is based on the observation that derivatives of the true solution are continuous across element faces while those of the finite element solution are discontinuous. For the Poisson problem in the one-dimensional case, the error is measured as follows:

$$\|E\|^2 = \int_\Omega \left(\frac{\partial U_{\text{ex}}}{\partial x} - \frac{\partial U_h}{\partial x}\right)^2 \mathrm{d}x.$$

The exact derivative is replaced by the approximation

$$\frac{\partial U_{\text{ex}}}{\partial x} \approx \left.\frac{\partial U}{\partial x}\right|^*$$

so that the error estimate is computed as

$$\|e\|^2 = \int_\Omega \left(\left.\frac{\partial U_{\text{ex}}}{\partial x}\right|^* - \frac{\partial U_h}{\partial x}\right)^2 \mathrm{d}x.$$

This approach will work provided that the recovered derivative

$$q^* = \left.\frac{\partial U}{\partial x}\right|^*$$

is more accurate than the finite element derivative q_h. Such approximations can be obtained by solving a least-squares problem [9,10]:

$$\text{Min } J(q^*) = \int_{\Omega_P} (q^* - q_h)^2 \, \mathrm{d}A,$$

where Ω_P is the whole domain for a global projection [9] or the subdomain consisting of elements surrounding node P for a local projection [10]. The local projection techniques have been shown to be more accurate and reliable than their global counterpart. The following polynomial expansion is used:

$$q^* = [1, x, y, x^2, xy, y^2][a_1, \ldots, a_6] = \boldsymbol{P}\vec{\boldsymbol{a}},$$

which leads to the 6×6 system for the coefficients:

$$\left[\int P^T P \, \mathrm{d}A\right]\{\boldsymbol{a}_i\} = \int P^T q_h.$$

Other least-squares variants are available, such as the superconvergent recovery [10]. The least-squares problem can be modified to accommodate additional terms ensuring satisfaction of the differential equations or boundary conditions [11].

Projection techniques are simple and have been termed crude but astonishingly effective [6]. Our experience indicates that they meet several of the previously listed criteria. They are general purpose and robust. They are blind to non-linearities, upwinding and smoothness of data. This is due to the fact that the finite element scheme used to solve the original problem does not enter into the construction of the error estimator.

For incompressible flow with heat transfer, estimators for each dependent variable are written as follows:

$$e_U^2 = \int (\boldsymbol{\tau}^* - \boldsymbol{\tau}_h) \colon (\boldsymbol{\tau}^* - \boldsymbol{\tau}_h) \, \mathrm{d}A,$$

$$e_P^2 = \int (p^* - p_h)^2 \, \mathrm{d}A,$$

$$e_T^2 = \int (\nabla T^* - \nabla T_h) \cdot (\nabla T^* - \nabla T_h) \, \mathrm{d}A,$$

with $\boldsymbol{\tau}$ being the deviatoric stress tensor

$$\boldsymbol{\tau} = \mu \begin{bmatrix} 2\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} & 2\dfrac{\partial v}{\partial y} \end{bmatrix}.$$

Note that velocity is considered as a vector unknown so that only one estimator is associated to it. Six local projections are performed: three for velocity, one for pressure and two for the temperature gradient. The projection matrix is the same for all projections. Only the right-hand-sides differ. Significant computational savings occur if the matrices are factored once and its factors reused as the need arises. For two-equation models of turbulence, six additional projections are required: four for the gradients of turbulence variables and two for the eddy viscosity gradient.

The issue of measuring errors from several sources must be tackled when solving such multi-field problems. This is a delicate issue since the errors for the various solution fields usually have different units. Using dimensionless equation constitutes a partial fix for two reasons. First, in many cases simulations are performed using dimensional numerical models. Second, non-dimensionalization does not always account for large differences in the magnitude of the dependent variables. For example, a dimensionless velocity field will be of first-order while the corresponding field of turbulence kinetic energy will generally be three to four orders of magnitude smaller.

A simple way of dealing with this problem consists in using the adaptive strategy to compute an ideal element sizes for each component of the solution (velocity, pressure, temperature, etc.). In the case of turbulent heat transfer, six ideal element sizes are derived for each element in the mesh. The smallest predicted element size is retained.

4. APPLICATIONS

This section presents a few applications of the methodology to turbulent flows. To set ideas, consider the k–ϵ model for incompressible turbulent flow. Such flows present a special challenge for CFD algorithm. They must preserve positivity of turbulence variables and of the eddy viscosity throughout the domain and through the course of iterations. Failure to do so results in dramatic breakdown of the solver. Limiters are often applied to various terms to ensure positivity. However, they slow down convergence, introduce noise in the solution, and destroy residuals. While this is a satisfactory approach for single-grid computations, it leads to

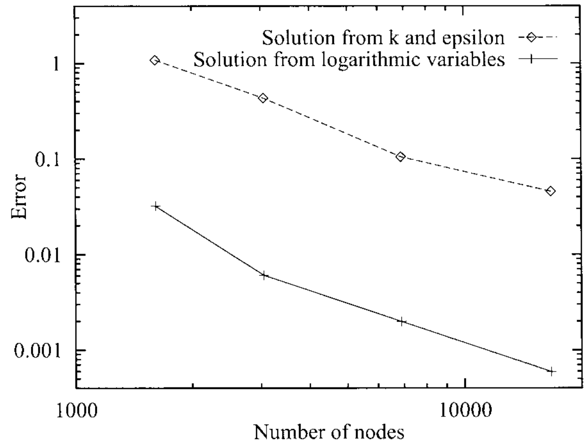

Figure 3. Accuracy improvement due to logarithmic variables.

difficulties in an adaptive context. First, a fairly fine initial mesh is usually necessary to ensure convergence. Second, adaptation is often performed either on noise or on the wrong flow features. Finally, in some cases it proves impossible to converge on the adapted mesh.

Many of these difficulties disappear if the natural logarithms of turbulence variables are used as computational variables. This is referred to as solving for *logarithmic variables* [13]. This amounts to the following change of dependent variables

$$K = \ln(k), \qquad E = \ln(\epsilon).$$

The turbulence model remains unchanged, while positivity of turbulence variables and eddy viscosity is guaranteed. Using logarithmic variables results in increased robustness of the solver, increased smoothness of the solution and dramatically improved accuracy of the turbulence field. This is especially true in regions of low turbulence. This is due to the fact that the logarithm varies more slowly than its argument. Logarithmic variables seldom show variations by more than a factor of ten in the domain while turbulence variables will often vary by six to eight orders of magnitude.

Figure 3 illustrates the dramatic improvements in accuracy of the eddy viscosity due to the use of logarithmic variables. The problem solved is a shear layer for which the eddy viscosity is a linear function of x only. Logarithmic variables improve accuracy of the eddy viscosity by a factor of 150! More on this topic may be found in [12–15].

Another advantage results from using logarithmic variables. Transport equations for other two-equation models can be obtained by linear combinations of those for K and E. This implies that a universal algorithm can be developed to treat all two-equation models with the same numerical technique. This eases implementation of different models and their comparison. See [16,17] for details and examples.

Figure 4 shows the initial and adapted meshes for the backward-facing step of Kim *et al.* [1]. Notice the extreme coarseness of the initial mesh. The final meshes for three models (k–ϵ, k–ω and k–τ) show refinement along the walls and in the shear layer emanating for the corner. Refinement in this region is due to the rapid changes in the eddy viscosity. Differences in the details of mesh refinement are most likely caused by differences in the turbulence models. See [16,17] for more details and further case studies.

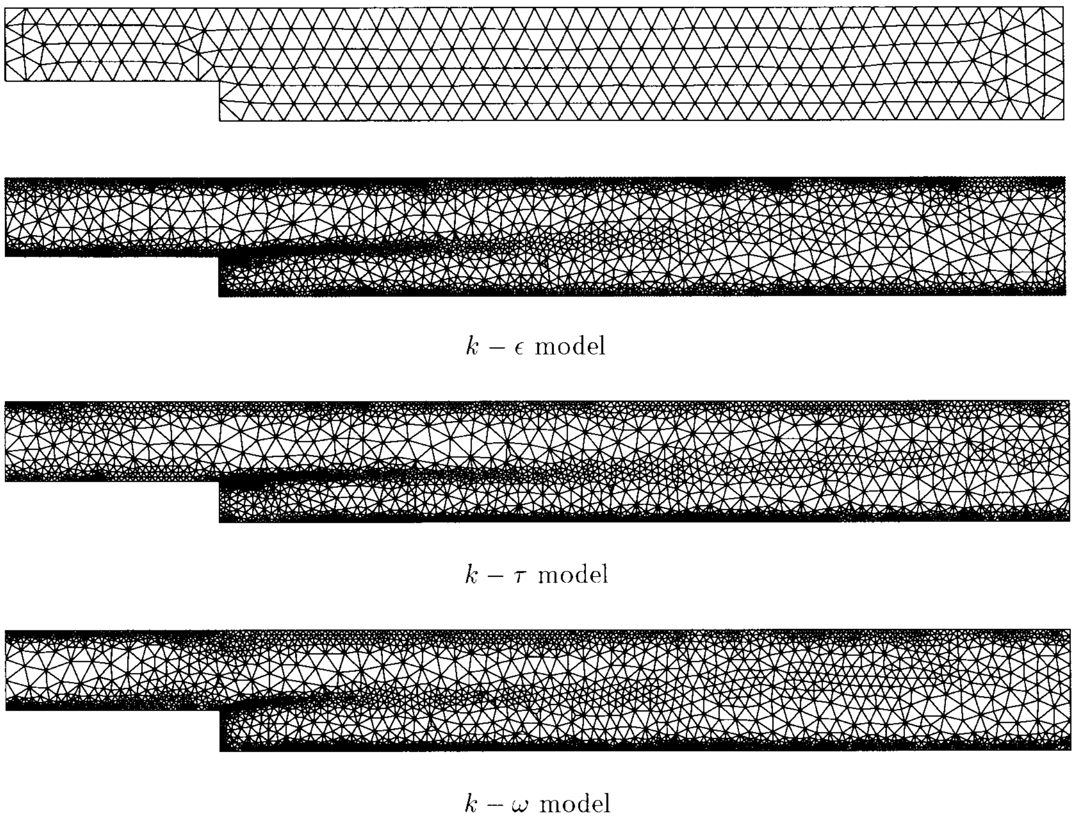

Figure 4. Meshes for step flow.

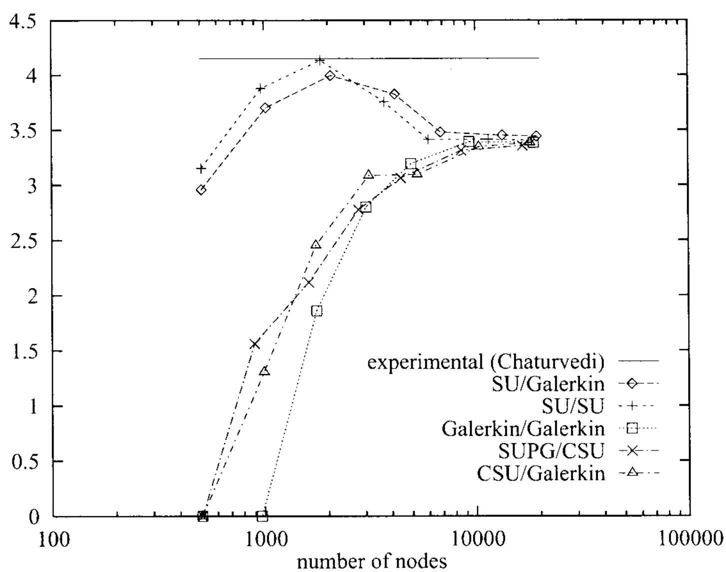

Figure 5. Recirculation length in conical diffuser.

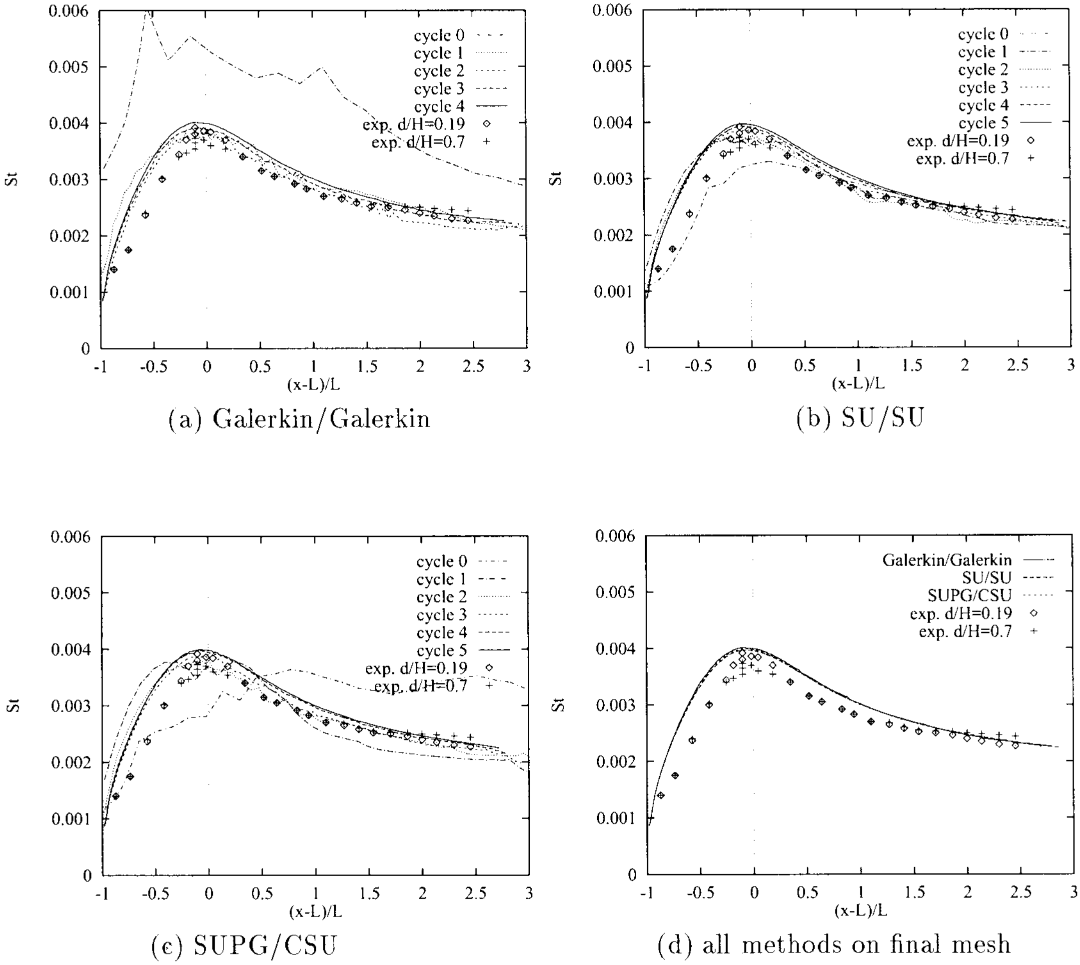

Figure 6. Stanton number predictions for the heated backward-facing step.

An adaptive mesh refinement study for turbulent flow in a conical diffuser was performed using different upwind finite element schemes. Figure 5 shows that the predicted length of the recirculation zone varies tremendously on the initial meshes. Some schemes even predict no recirculation. However, as the mesh is refined predictions by all schemes converge to the same value. Note that grid-independent predictions are achieved only on relatively fine meshes. The behaviour of the streamline–upwind method (SU) should serve as a reminder that apparent agreement with measurements can be misleading without careful grid refinement studies. Details are available in [14]. Figure 6 shows results from adaptive grid refinement studies for turbulent forced convection over a backward-facing step [18]. Adaptivity leads to grid- and scheme-independent predictions of the Stanton number. Further details may be found in [14].

Finally, results for turbulent flow over an RAE-2822 airfoil are shown in Figure 7. Computations were performed at a Mach number of 0.725 and an angle of attack of 4° using the k–ϵ model. Grid converged results are obtained after three cycles of mesh adaptation. Note that the numerical shock is not as diffused as the measured one. This may have two causes. First, measurements are taken on the airfoil surface inside the sublayer. In this region,

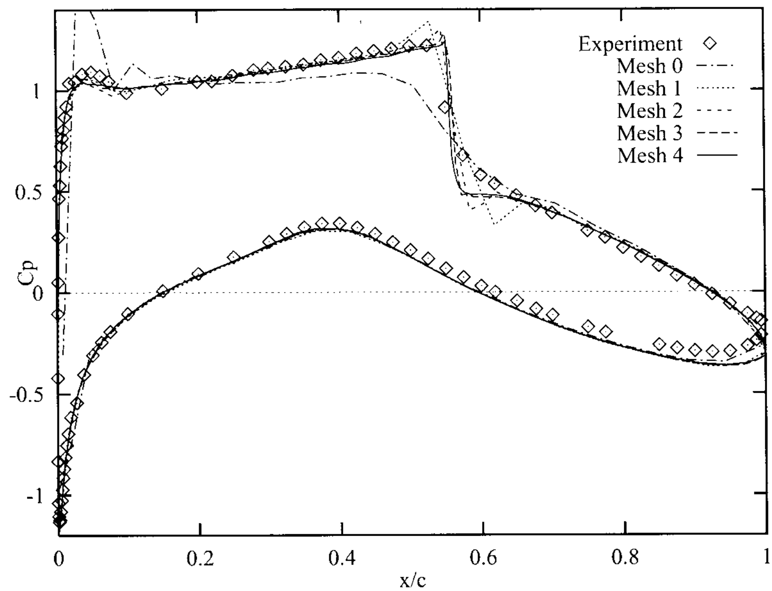

Figure 7. Turbulent flow prediction over REA-2228 airfoil.

the Mach number is very low so that pressure waves can travel upstream. In fact, the shock does not reach the airfoil surface due to the boundary layer. Secondly, the simulation uses wall functions that do not account for the viscous sublayer. Details may be found in [19].

Finally, interested readers will find applications of the methodology to laminar flows with various error estimators in [20,21], conjugate heat transfer in [22], heat transfer by forced, mixed and free convection in [23–25], turbulent flows and heat transfer in [26–30], compressible and incompressible flows in [31], optimal design of fluid flow system in [32], flows with free surfaces and surface tension in [33,34].

5. CONCLUSION

We have provided a brief survey of error estimation and its application to adaptive solution of complex flows. Several sources of errors in a computational model have been identified.

Experience indicates that projection techniques are remarkably robust. At this time they seem to offer an excellent compromise for general purpose application. They have been successfully applied to a broad spectrum of flows.

Finally, this paper has shown how the technique can be used to perform rigorous adaptive grid refinement studies to verify computations, achieve grid-independent results and compare mathematical models of turbulence or different upwind discretization techniques.

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REFERENCES

1. J. Kim, S. Kline and J. Johnston, 'Investigation of separation and reattachment of a turbulent shear layer, flow over a backward facing step', *Report MD.37*, Department of Mechanical Engineering, Stanford University, 1978.
2. M. Nallasamy, 'Turbulence models and their applications to the predictions of internal flows', *Comput. Fluids*, **15**, 151–194 (1987).
3. B.E. Launder, 'Current capabilities for modelling turbulence in industrial flows', *Appl. Sci. Res.*, **48**, 247–269 (1991).
4. I. Babuška and W. Gui, 'Feedback, adaptivity and *a posteriori* estimates in finite elements: aims, theory, and experience', in I. Babuška, J. Gago and A. Oliveira (eds.), *Accuracy Estimates and Adaptive Refinements in Finite Element Computations*, Wiley, Chichester, 1986, p. 393.
5. J. Perarire, M. Vahdati, K. Morgan and O.C. Zienkiewicz, 'Adaptive remeshing for compressible flow computations', *J. Comput. Phys.*, **72**, 26–37 (1987).
6. M. Ainsworth and J.T. Oden, '*A posteriori* error estimation in finite element analysis', *Comput. Methods Appl. Mech. Eng.*, **142**, 1–88 (1997).
7. A. Blain, J.J. Westerink and R.A.J. Luettich, 'Grid refinement studies for the prediction of hurricane storm surge', *Int. J. Numer. Methods Fluids*, **26**, 369–404 (1998).
8. H.H. Dannelongue and P.A. Tanguy, 'An adaptive remeshing technique for non-Newtonian fluid flow', *Int. J. Numer. Methods Eng.*, **30**, 1555–1567 (1990).
9. O.C. Zienkiewicz and R. Zhu, 'A simple error estimator and adaptive procedure for practical engineering analysis', *Int. J. Numer. Eng.*, **24**, 337–357 (1987).
10. O.C. Zienkiewicz and J.Z. Zhu, 'The superconvergent patch recovery and *a posteriori* error estimators. Part 1. The recovery technique', *Int. J. Numer. Methods Eng.*, **33**, 1331–1364 (1992).
11. N.-E. Wiberg, F. Abdulwahab and S. Ziukas, 'Enhanced superconvergent patch recovery incorporating equilibrium and boundary conditions', *Int. J. Numer. Methods Eng.*, **37**, 3417–3440 (1994).
12. F. Ilinca, D. Pelletier and A. Garon, 'An adaptive finite element method for a two-equation turbulence model in wall-bounded flows', *Int. J. Numer. Methods Fluids*, **24**, 101–120 (1997).
13. F. Ilinca and D. Pelletier, 'Positivity preservation and adaptive solution for the k–ϵ model of turbulence', *AIAA J.*, **36**, 44–51 (1998).
14. E. Turgeon and D. Pelletier, 'Effects of adaptivity on various finite element schemes for turbulent heat transfer and flow predictions', *36th AIAA Aerospace Sciences and Exhibit*, *Paper AIAA-98-0229*, 12–15 January, 1998.
15. E. Turgeon and D. Pelletier, 'Adaptive computation of impingement heat transfer', *AIAA/ASME Joint Thermophysics Conference*, Albuquerque, NM, 15–18 June 1998.
16. L. Ignat, D. Pelletier and F. Ilinca, 'Solution of two-equation models of turbulence using a universal adaptive finite element algorithm', *36th AIAA Aerospace Sciences and Exhibit*, *Paper AIAA-98-9765*, 12–15 January, 1998, pp. 33–40.
17. F. Ilinca, J.-F. Hétu and D. Pelletier, 'A single formulation and finite element algorithm: a tool for comparing two-equation models of turbulence', *AIAA 97-2073*, 1997.
18. J.C. Vogel and J.K. Eaton, 'Combined heat transfer and fluid dynamic measurements downstream of a backward-facing step', *J. Heat Transf.*, **107**, 922–929 (1985).
19. F. Ilinca, D. Pelletier and L. Ignat, 'Adaptive finite element solution of compressible turbulent flows', *Paper AIAA 98-0229*, 1998.
20. J.-F. Hétu and D. Pelletier, 'Adaptive remeshing for viscous incompressible flows', *AIAA J.*, **30**, 1986–1992 (1992).
21. J.-F. Hétu and D. Pelletier, 'Fast, adaptive finite element scheme for viscous incompressible flows', *AIAA J.*, **30**, 2677–2682 (1992).
22. D. Pelletier, L. Ignat and F. Ilinca, 'Adaptive finite element method for conjugate heat transfer', *Numer. Heat Transf. Part A*, **32**, 267–289 (1997).
23. D. Pelletier, J.-F. Hétu and F. Ilinca, 'Adaptive finite element method for thermal flow problems', *AIAA J.*, **32**, 741–747 (1994).
24. D. Pelletier, F. Ilinca and J.-F. Hétu, 'An adaptive finite element method for convective heat transfer with variable fluid properties', *AIAA 93-2767*, 1993.
25. D. Pelletier and F. Ilinca, 'An adaptive finite element method for mixed convection', *AIAA J. Thermophys. Heat Transf.*, **9**, 708–714 (1995).

26. L. Ignat, D. Pelletier and F. Ilinca, 'An adaptive finite element method for turbulent heat transfer', *AIAA 34th Aerospace Sciences Meeting and Exhibit*, *AIAA Paper 96-0607*, Reno, NV, January 15–18, 1996.
27. D. Pelletier, F. Ilinca and J.-F. Hétu, 'An adaptive finite element method for turbulent free shear flow near a propeller', *AIAA J.*, **32**, 2186–2193 (1994).
28. F. Ilinca, D. Pelletier and F. Arnoux-Guisse, 'An adaptive finite element scheme for turbulent free shear flows', *Int. J. CFD*, **8**, 171–188 (1997).
29. D. Pelletier and F. Ilinca, 'Adaptive remeshing for the $k-\epsilon$ model of turbulence', *AIAA J.*, **35**, 640–646 (1997).
30. F. Ilinca, D. Pelletier and A. Garon, 'An adaptive finite element method for a two-equation turbulence model in wall bounded flows', *Int. J. Numer. Methods Fluids*, **24**, 101–120 (1997).
31. F. Ilinca and D. Pelletier, 'A unified approach for adaptive solutions of compressible and incompressible flows', *AIAA 35th Aerospace Sciences Meeting and Exhibit*, *AIAA Paper 97-0330*, Reno, NV, January 6–10, 1997.
32. J. Borggaard and D. Pelletier, 'Optimal shape design in forced convection using adaptive finite elements', *36th AIAA Aerospace Sciences and Exhibit*, January 12–15, *Paper AIAA-98-0908*, 1998, pp. 33–40.
33. S. Dufour and D. Pelletier, 'An adaptive finite element method for multiphase flows with surface tension', *IVth World Congress on Computational Mechanics*, Buenos Aires, Argentina, June 29–July 2, 1998.
34. S. Dufour and D. Pelletier, 'An adaptive finite element method for incompressible flows with interfaces', *29th AIAA Fluid Dynamics Conference*, Albuquerque, NM, 15–18 June 1998.

